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ELASTI-PLASTIC DESIGN OF SINGLE-SPAN
BEAMS AND FRAMES

by Herbert A. Sawyer, Jr., M. ASCE

STRUCTURAL DIVISION

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THIS PAPER

ELASTIC-PLASTIC DESIGN OF SINGLE-SPAN
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Herbert A. Sawyer, Jr.,[*] M. ASCE

SYNOPSIS

A static-deformational analysis shows that, in general, conventional limit design violates either statics or the deformational limitations of a structural material in varying degrees, and that, therefore, limit design is unreliable in obtaining collapse loads.

After the development of simple expressions for both elastic and plastic flexural deformations, a general, practicable procedure for the solution of restrained single-span beams and frames is presented which makes maximum use of familiar procedures of elastic analysis. The use of this method on beams and frames is illustrated, using moment-curvature data for reinforced concrete members obtained at the University of Connecticut.

Results of the elasti-plastic analysis of several structures are compared with the results of the conventional elastic and plastic analyses of the same structures. Finally, possible sources of error for the method are briefly evaluated, including inherent limitations of the method and factors other than moment which affect curvatures.

Limitations of Present Theories

Although the resultant forces in a simple, statically determinate structure are determined by the law of statics, the forces in a redundant structure are determined by both statics and the stress-strain relationship for the material. For a structure consisting of flexural members, the stress-strain relationship is best represented by a moment vs. curvature diagram, a typical example of which is Fig. 1(a). As shown by this figure, the actual relationship between bending moment, M, and curvature, ϕ, for a flexural structural member usually represents a combination of elasticity and plasticity.

The classical elastic analysis of a structure is based on a linear M-ϕ relationship, Fig. 1(b), and neglects the plastic portion of this relationship. Therefore, as is well known, an elastic analysis will only give correct force and deflection results if no moment in the actual structure exceeds the elastic limit, and an elastic analysis is adequate only for the structure for which fatigue or the avoidance of permanent set governs. Also, elastic analysis is reasonably accurate for the variable strength structure proportioned so that elastic strength closely matches the imposed maximum moment at all sections. For designing against failure from non-fatigue fracture or plastic buckling an elastic analysis is generally incorrect.

On the other hand, the recently developed plastic or limit analysis,[1] first given formal prominence in this country by van den Broek,[2] assumes that the moment corresponding to any curvature is the yield moment, M_y, Fig. 1(c), thus neglecting any elastic curvature. This omission, plus the indeterminacy

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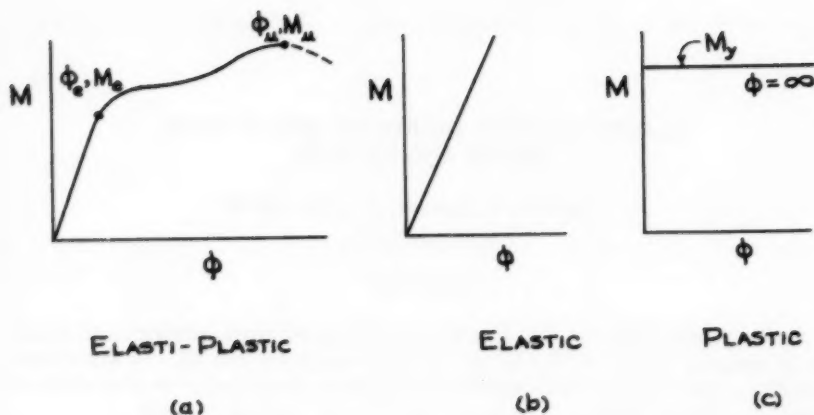[*] Associate Prof. of Civ. Eng., Univ. of Connecticut, Storrs, Conn.

Fig. 1 Moment-curvature relationships

of the relationship between the plastic moment and plastic curvature, indicates the weakness of limit analysis: it neglects any limitation on loading connected with limitation of deformational quantities. A limit analysis only determines the load and limit moments at which the rate of increase of deflection with respect to increase in load becomes infinity, $\frac{d\Delta}{dP} = \infty$. Actually, many structures will fail before all the plastic hinges required for kinematic instability have been developed because of one of these deformational limitations:

(a) Excessive deflections (steel or reinforced concrete)
(b) Fracture (usually concrete)
(c) Buckling (usually steel)

This deficiency is emphasized by the free bodies, shown in Fig. 2, of the plastic region of a beam supporting a concentrated load. If the moments acting on these free bodies are in equilibrium:

$$M_u - M_e = VS_L \qquad M_u - M_e = (P-V)S_R \qquad (1\,a,b)$$

If the plastic moments in these equations are equal ($M_u = M_e$), as assumed by the plastic theory, S_L and S_R must be zero, and the plastic zone and all plastic bending is concentrated at the load. Thus the curvature at this concentration is infinite, causing failure in any known material.

The moment diagram for this length of beam illustrates the same static requirements. By statics the slope at any point on the moment diagram must equal the shear. In general the shears and hence slopes to either side of the load cannot equal zero, and therefore the plastic moment M_y can exist at only one point, the point of loading, again proving infinite plastic curvature at this point.

Of course every real load is somewhat distributed. Also, for most members the moment-curvature relationship shows some strain hardening and increase in moment in the plastic range of curvatures. If M_u is the ultimate moment, this increase in moment will be $M_u - M_e$, and from the free bodies of Fig. 2 the length of the plastic zone is, by statics

$$S_L + S_R = (M_u - M_e)\left(\frac{1}{V} + \frac{1}{P-V}\right) \qquad (2)$$

Thus, as Hrennikoff[3] has previously implied, the rotational capacity of a plastic hinge is somewhat proportional to the strain-hardening moment, $M_u - M_e$, as well as the ultimate plastic curvature ϕ_u. These quantities, neglected by the plastic theory, are of primary importance in determining a structure's capacity for plastic behavior.

Knudsen et al,[4] Symonds and Neal,[5] and Ernst,[6] have presented methods for calculations of deflections for limit-designed structures on the assumption that the structures can take the limit-analysis moments. These methods are intended to furnish a check against failure from source (a) above, excessive deflections, but not failure from sources (b) or (c).

Also, the structural strengths given by the elastic and plastic theories are usually quite different, and the engineer should not be forced to choose between these extremes in designing actual structures made of materials which are neither perfectly elastic nor perfectly plastic.

A theory is needed which will provide a compromise between the extremes of the elastic and plastic theories—a theory based on the deformational capacity of the structural material as determined by: (1) statics, and (2) the
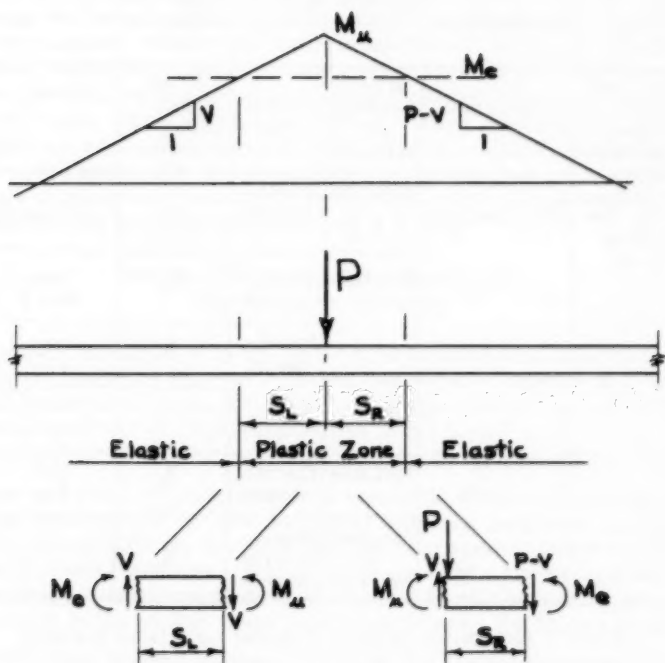

Fig. 2 Static requirements for a plastic hinge

material's elastic-plastic moment-curvature relationship. Such a theory may be termed an "elastic-plastic theory". Cross[21] and others have suggested procedures, and Hrennikoff[3] has formulated many of the basic relationships for such a theory. However, the application of these procedures and relationships to actual structures has been of forbidding complexity. The aim of the remainder of this paper is to state basic principles in the most simple terms and to present analytical methods which will minimize the complexities and labor to an elasti-plastic solution, making it a more practicable tool for the structural engineer.

The proper functions and limitations of the elastic theory, plastic theory (limit analysis), and the elasti-plastic theory for flexural structural design are shown in Table I. The reliability of these theories for each class of design is somewhat roughly indicated in the last column by the ratio of the calculated to the actual maximum load, unity obviously being the ideal value for this ratio.

Design Criteria	Statically	I	Moments Obtained by	Maximum Allowable Moment	Range of Ratio: Calc. Max. Load / Actual Max. Load
Maximum load before either: (a) Permanent set (b) Fatigue failure	Determinate	Either constant or variable	Statics only	$M_e = \frac{\sigma I}{c}$ σ = el. limit or endurance limit	1
	Indeterminate		Elastic theory		1
Maximum load before collapse, neglecting limitations on deflections, and with few repetitions of load	Determinate	Either constant or variable	Statics only	M_u by plastic theory or tests	1
	Indeterminate	Variable $I = \frac{Mc}{\sigma}$	Elastic theory	M_u	1 (approx.)
		Constant	Elastic theory	M_u	from less than ½ to 1
			Plastic theory	M_u	from 1 to 2 or more
				M_e	(from 1 to 2 or more) $\left(\frac{M_e}{M_u}\right)$
		Variable	Plastic theory	M_u	from 1 to 2 or more
		Constant or variable	Elasti-plastic theory	M_u	1

Table I Evaluation of basic flexural structural theories

For those designs for which the elastic theory is inaccurate, this table shows the general superiority of the elasti-plastic theory over the plastic theory. Obviously, this superiority decreases as the capacity for plastic

curvature increases, causing the ratio of calculated to actual maximum load to approach unity for the plastic theory. Admittedly, the comprehensiveness and brevity of this table are attained at the expense of oversimplification. (Freudenthal[26] has ably discussed in greater detail the relationship between the elastic and plastic theories.)

The Moment-Curvature Relationship

This paper does not consider the problem of obtaining, either by theory or experiment, moment-curvature relationships; it is confined to presenting a practical method of using any given moment-curvature relationship for an elasti-plastic structural analysis. (Factors which may prevent a given moment-curvature relationship from applying to an actual structural situation will be discussed later.) Extensive information on moment-curvature relationships may be found in the following references.

For a theoretical investigation of the short-term moment-curvature relationship for steel members, reference may be made to Timoshenko,[7] Winter,[8] and Hrennikoff.[3] The more recent work of Johnston et al[9, 10, 11, 12] is outstanding both experimentally and theoretically on this problem. Clark, Corten, and Sidebottom[13] have made a thorough experimental and theoretical investigation of M-ϕ relationships for steel and alloy members under both short-term and sustained loads. Ketter, Kaminsky, and Beedle[14] have investigated the effect of residual stresses and axial load on the M-ϕ relationship.

Most of the investigations of reinforced concrete have been concerned with the stresses and strains at a section of a member subjected to its ultimate short-term moment, the purpose being to provide a theory for ultimate strength at a section, recent examples being the work of Chambaud[15] and Hognestad.[16] Lee[17] has investigated the behavior of reinforced concrete members under short-term loads. Experimental moment-curvature relationships have been determined in a recently completed project[18] of the Civil Engineering Department at the University of Connecticut for both short-term and sustained loads. One of these experimentally-determined relationships will be used for the illustrative solutions which follow.

General Evaluation of the Elastic and
Plastic Angle Changes

A necessary preliminary to the elasti-plastic solution of a flexural structure is the evaluation of the elasti-plastic flexural deformation in a length of beam, this deformation being a function of a given moment-curvature relationship and the shear-moment distribution in the beam.

It will be assumed that for the length of beam being considered:

1. Shear and axial <u>deformations</u> have negligible effect on the solution. (It is not necessary to assume that shear and axial load have negligible effect on curvatures; the M-ϕ diagram may be adjusted for such effects.)
2. The moment-curvature relationship is constant. The factors which tend to make this relationship vary along a length will be mentioned later.
3. Moments in the plastic range do not decrease as the loads are applied.

It is convenient to consider the curvatures of a typical moment-curvature diagram such as Fig. 3(a) as being made up of the two parts shown, the "elastic" and the "plastic". Then, if the moments in a portion of a member are as shown in Fig. 3(b), where M_m, the maximum moment in the portion, has a

value between M_e and M_u, the resulting curvatures throughout the portion may be obtained, Fig. 3(c), the curvature at a point where the moment is M being $\phi = \phi_E = \phi_P$. Then since

$$d\theta_E = \phi_E \, ds = \frac{M}{M_e} \phi_e \, ds \qquad (3)$$

the relative elastic rotation of the beam axis at a point A with respect to the axis at a point B would be

$$(\theta_{A/B})_E = \frac{\phi_e}{M_e} \int_B^A M \, ds \qquad (4)$$
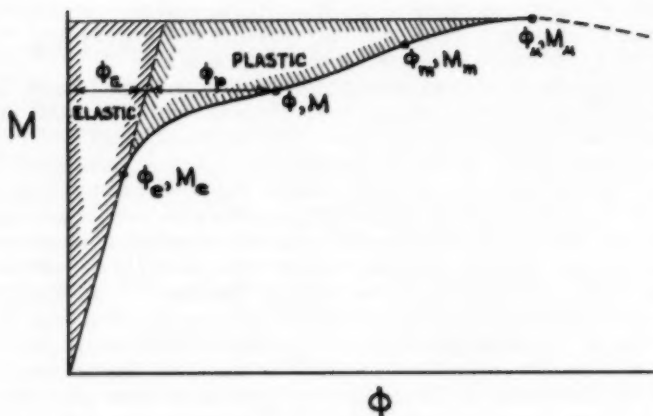
TYPICAL MOMENT-CURVATURE DIAGRAM

(a)

Fig. 3(a) Elasti-plastic curvature

As would be expected, this expression corresponds to the familiar first moment-area theorem for elastic deformations if the constant, $1/EI$, is replaced by the equivalent constant, ϕ_e/M_e. Thus, the familiar moment-area (or conjugate-beam) theorems and techniques, including the principle of superposition, may be used for evaluating the components of deflection from the elastic curvature.

Similarly, the total plastic angle of relative rotation is:

$$\theta_p = \int^{S_p} \phi_p \, ds = \int^{S_p} (\phi - \frac{M}{M_e} \phi_e) \, ds \qquad (5)$$

this plastic angle merely being the "plastic" area of the curvature diagram, Fig. 3(c). This area may be plotted and evaluated for any known moment-curvature diagram and moment diagram. However, the calculation of these

plastic angles occurs so often in an elasti-plastic analysis that more efficient methods of evaluation are desirable.

TYPICAL MOMENT DIAGRAM
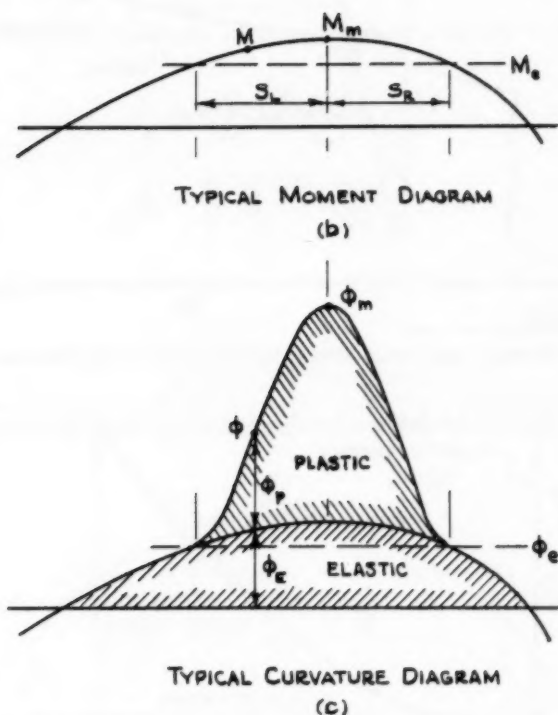
(b)

TYPICAL CURVATURE DIAGRAM

(c)

Fig. 3(b, c) Elasti-plastic curvature (concluded)

The most common moment diagrams are those for a concentrated or uniformly distributed load, diagrams which consist of straight lines or a second degree parabola, respectively. For either of these types the ratio of plastic angle area to the plastic length, θ_p/S_p, may be plotted against the ratio of M_m to M_e. From this diagram the plastic angle, θ_p, may be easily obtained.

However, for practical purposes the two-segment idealization of a moment-curvature diagram shown in Fig. 4(a) is always accurate enough, compared to the accuracy of other basic assumptions. Of course the line from M_e to M_u should be placed so as to define a plastic area which closely approximates the actual area in magnitude and centroid, remembering that an under-estimation of the plastic area is always on the side of safety. The small loss in accuracy for an idealized diagram is shown in Fig. 4(b), which shows values of the plastic angle for an actual and an idealized moment-curvature diagram, as well as values for the conventional elastic and limit analyses.

Analytical evaluation of the plastic angles for the idealized moment-curvature diagram of Fig. 4(a) and the common linear or parabolic moment diagrams results in simple and usable expressions. In general, using the similar triangles of Fig. 4(a):

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ACTUAL AND IDEALIZED MOMENT-CURVATURE DIAGRAM
(a)

COMPARISON OF PLASTIC ANGLES FOR TWO-SEGMENT MOMENT DIAGRAM
(b)

TWO-SEGMENT MOMENT DIAGRAM
(c)
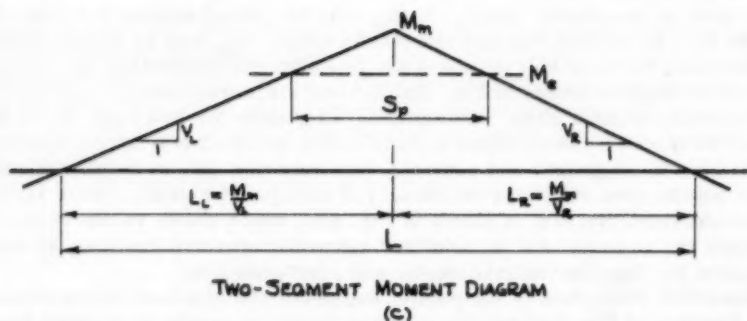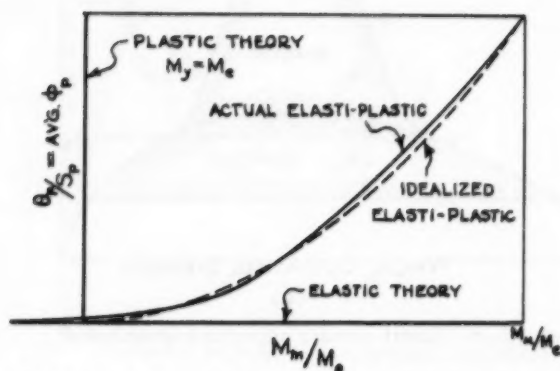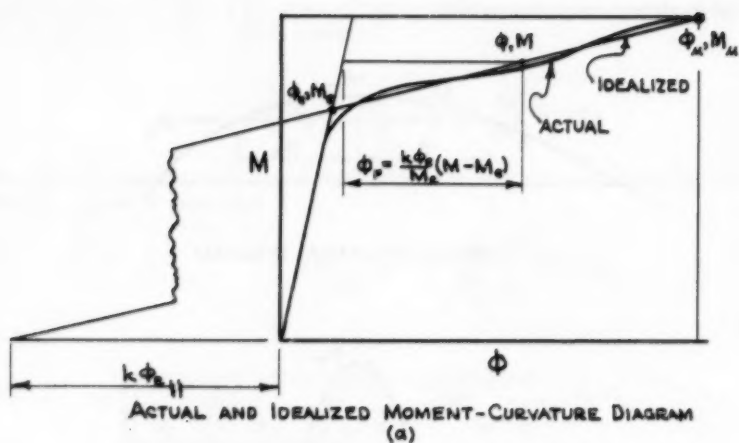
Fig. 4 Approximation of moment-curvature diagram

$$\theta_p = \int \phi_p \, ds = \frac{k\phi_e}{M_e} \int^{S_p} (M - M_e) ds = \frac{k\phi_e}{M_e}(Plastic\text{-}Moment\ Area) \tag{6}$$

Evaluating this expression for the linear moment diagram of Fig. 4(c):

$$\theta_p = k\frac{\phi_e}{M_e}\frac{(M_m - M_e)}{2}S_p$$

$$\theta_p = \frac{k}{2}\frac{\phi_e}{M_e}\frac{(M_m - M_e)^2 L}{M_m} \tag{7}$$

where L is the length between points of zero moment, Fig. 4(c).
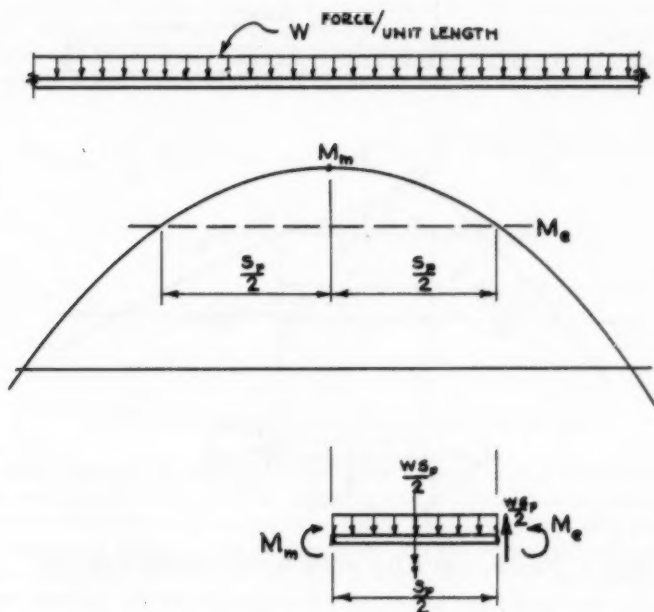
Fig. 5 Plastic region for uniform load

Evaluating the general expression, Eq. 6, for the second degree parabolic moments of Fig. 5:

$$\theta_p = k\frac{\phi_e}{M_e}\frac{2}{3}S_p(M_m - M_e)$$

But from the free body of Fig. 5:

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$$M_m - M_e = \frac{w S_p^2}{8}$$

Hence:

$$\theta_p = 1.886 \, k \, \frac{\phi_e}{M_e} \sqrt{\frac{(M_m - M_e)^3}{w}} \qquad (8)$$

Since superposition cannot be used for computation of the plastic angle, a flexural member must often be analyzed for a combination loading consisting of a concentrated load or reaction and a distributed load. For this case, referring to Fig. 6:
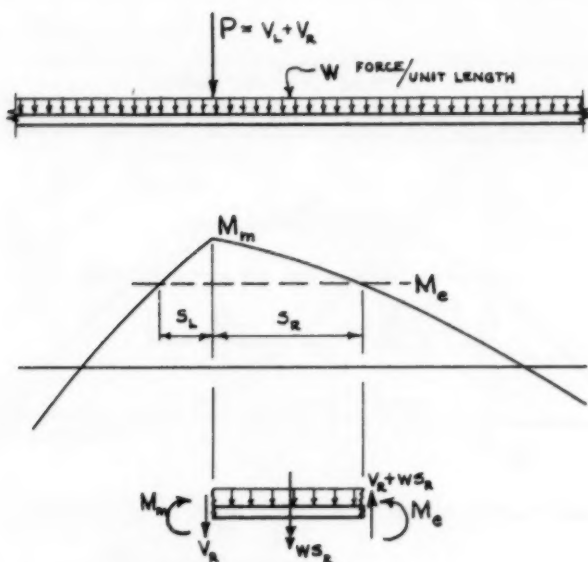
Fig. 6 Plastic region for uniform and concentrated load

$$\theta_{pR} = k \, \frac{\phi_e}{M_e} \, \frac{S_R}{2} \left(M_m - M_e + \frac{w S_R^2}{6} \right) \qquad (9)$$

and from the free body of Fig. 6:

$$S_R = \sqrt{\left(\frac{V_R}{w} \right)^2 + \frac{2(M_m - M_e)}{w}} - \frac{V_R}{w} \qquad (10)$$

where θ_{pR} is the plastic angle from moments to the right of the concentrated

load, and V_R is the beam shear to the immediate right of the concentrated load. Expressions for the plastic angle to the left, θ_{pL}, and S_L are the same as Eqs. 9 and 10 with all subscripts R replaced by L. Of course the total plastic angle will be:

$$\theta_p = \theta_{pR} + \theta_{pL} \qquad (11)$$

Actually, for this combination case Eq. 7 may usually be applied with sufficient accuracy provided that the distributed load is relatively small and that L is taken as:

$$L = L_L + L_R = \frac{M_m}{V_L} + \frac{M_m}{V_R} \qquad (12)$$

If the term M_e/ϕ_e (or EI) is constant throughout a structure, it may be omitted from both numerator and denominator of the fractions used to calculate the redundant forces. Using a prime superscript (') to indicate angles or displacements from which this term has been omitted, the following simplified expressions result from Eqs. 4, 7, 8, and 9, respectively:

$$(\theta_{A/B})_E' = \int_B^A M\,ds \qquad \theta_p' = \frac{k}{2}\frac{(M_m - M_e)^2 L}{M_m} \qquad (13, 14)$$

$$\theta_p' = \frac{1.886k}{w^{\frac{1}{2}}}(M_m - M_e)^{\frac{3}{2}} \qquad (15)$$

$$\theta_p' = \frac{kS_B}{2}\left(M_m - M_e + \frac{w S_B^2}{6}\right) \qquad (16)$$

Solutions of Single-Span Restrained Structures

An elasti-plastic solution of a statically indeterminate flexural structure may be performed, in theory, at least, by the solution of a number of equations equal to the degree of indeterminacy, these equations being of a degree equal to the number of plastic regions plus one. To add to the difficulties inherent to such a solution, the number of plastic regions is: (1) not known before the analysis, and (2) not restricted to the degree of indeterminacy plus one, as for a limit analysis. Thus such a solution is at best prohibitively tedious. This paper simplifies an elasti-plastic solution, not only by use of the successive-approximation type of approach previously used so successfully by Cross[19] and others for complex problems and suggested by Cross[21] for this problem, but by the use of the formulas just derived and certain approximations and special techniques which will be described.

The general principle for the elasti-plastic (as well as elastic) solution of any indeterminate structure is that the correct static force system has been found if the structural deformations caused by these forces are consistent with the given restraints to deformation.

In accordance with this general requirement, each trial of a trial-and-error solution may consist of the following well-known steps:

1. Assume a static force system for the structure.
2. Calculate the elastic and plastic angle changes caused by these forces.
3. Calculate the deviation of the deflections caused by these angle changes from the given restraints.
4. Using these deviations as a guide, correct the assumed force system.

The assumed force system is best defined by a number of assumed forces equal to the degree of statical indeterminacy. Then the moment at any point may be expressed in terms of these forces using statics. Relationships expressed by using the moments at the suspected plastic hinges as variables are generally more complex because these moments generally exceed in number the degree of indeterminacy, necessitating use of an additional number (equal to the excess) of equations based on statics.

Also, the assumed forces are best considered as acting at the centroid of the $1/EI$ areas of the structure (the elastic center or neutral point) and in the direction of the principle axes of these areas. As is well known,[20] this procedure serves to eliminate all components of deflection from elastic angle changes except that component in the direction of the applied force.

The point of departure, or first trial, for an elasti-plastic analysis will be the forces found from the elastic solution, which will be sufficiently good as a first approximation to the final forces. Also, the elastic solution provides elastic properties of the structure which are useful for the elasti-plastic analysis.

In the following illustrative examples, three simplifying assumptions and approximations will be found. It should be noted that these assumptions and approximations are not inherent to the general procedure; that is, if they were not made, the same procedure could be used to solve the structure.

First, the term $\dfrac{M_e}{\phi_e}$ (or EI) is assumed constant throughout the structures, allowing use of Eqs. 13 to 16 in evaluating angle changes. Second, the plastic angle is assumed to be concentrated at the point of maximum moment. This assumption is true for the plastic angle of Eq. 15. It is true enough for practical purposes for the plastic angles of Eqs. 14 and 16, unless the ratio M_m/M_e or the distance L becomes relatively large. Third, as discussed previously, the moment-curvature relationship is assumed to be adequately represented by a typical two-segment diagram, Fig. 7, a necessity for use of Eqs. 13 to 16 in evaluating the plastic angles.
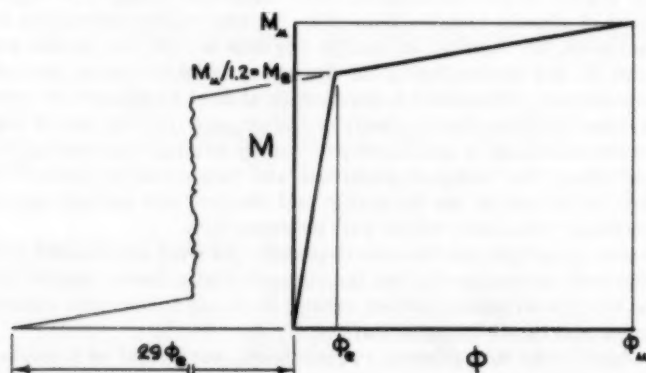

Fig. 7 M-ϕ diagram used for examples

The values of $k = 29$ and $M_u/M_e = 1.2$ of Fig. 7 approximate experimental results[18] for certain under-reinforced concrete beams under sustained loads. (The general use of these values for reinforced concrete would often be unsafe, and is not recommended.) The test beams were made of high strength concrete, cylinder strength averaging 5750 psi, with 2.1 percent intermediate grade steel of 46,000 psi yield point and with applied moment increased in the plastic range at the rate of $0.006 \, M_u$ per day until failure. The curvature at ultimate moment averaged .0036 radians per inch with a depth $\underline{d}$ of 4 in. Fig. 8 shows one of the test beams at practically its ultimate curvature.
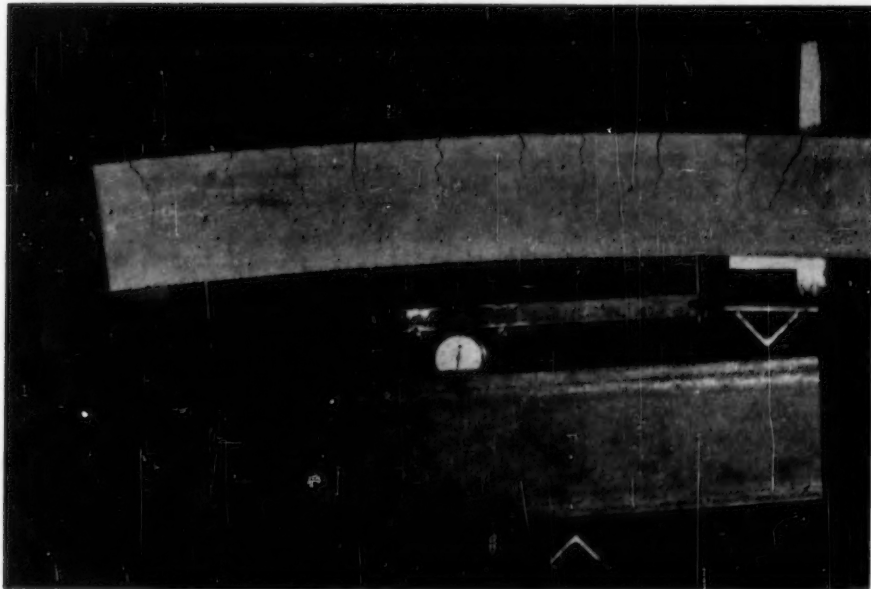
Fig. 8. Long-term-loaded reinforced concrete beam at near-ultimate curvature

Example 1—Design of Single-Span Restrained Beam

Referring to Fig. 9, the minimum bending strength required of member AC to support the 20 kip load located as shown will be determined. The dead load will be considered negligible, and the moment-curvature relationship shown in Fig. 7 will be used.

First, the elastic solution is obtained in the computations above the double line of Fig. 9, using the well-known elastic center or neutral point method.[20] (This method is numerically identical to the column analogy.[21] Statically indeterminate to the second degree, the beam may be cut at C, with the restorative moment at elastic center O, M_O, and the restorative vertical force at O, Y_O, used as the redundants. As usual, each of these redundants is determined as the ratio of the total displacement component of the elastic center, Fig. 9(b), to the displacement component from a unit moment or force. Having determined these redundants, Fig. 9(c), the moments in the beam are computed for all points of maximum moment, that is, points of initial plasticity
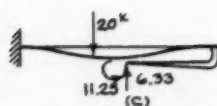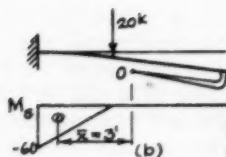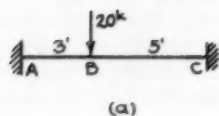
Sign Convention: Forces at 0, or movements of 0 $\curvearrowleft \downarrow \rightarrow$ +

(a) (b) (c)

S	$\int x_0^2 ds$	$\theta_0' = \int M_s ds$	$\bar{x}_0$	$\Delta_0' = \int x_0 M_s ds$
8	$8^3/12 = 42.7$	-90	-3	$+270$

$$M_A \quad M_B \quad M_C \quad M_e$$
$$(x_s = -4) \quad (-1) \quad (+4)$$

M_A	M_B	M_C	M_e
$-60.$	0	0	
$+11.25$	$+11.25$	$+11.25$	
$+25.32$	$+6.33$	-25.32	
-23.43	$+17.58$	-14.08	19.52

$$M_0 = -\frac{-90}{8} = +11.25^{k'} \qquad Y_0 = -\frac{+270}{42.7} \quad -6.33^k$$

				θ_0'	Δ_0'	
				0	0	a
$\theta_{pA}' = -14.5(1.72)3.91^2/23.43$				-16.2	$+64.8$	b,c
				$r = (-.33)$ (assume)	$(-.33)$	d ⎫
$+.68$	$+.68$	$+.68$		$\Delta_1 M_0 = \frac{+5.4}{8} = +.68$	$\Delta_1 Y_0 = \frac{-21.6}{42.7} = -.507$	e ⎬ 1
$+2.03$	$+.51$	-2.03				f
-20.72	$+18.77$	-15.43	17.27			g ⎭
				$+5.4$	-21.6	a
$\theta_{pA}' = -14.5(1.57)3.45^2/20.72$				-13.1	$+52.4$	b
$\theta_{pB}' = +14.5(1.43+2.75)1.50^2/18.77$				$+7.3$	-7.3	b
				$-.4$	$+23.5$	c ⎫
				$-\frac{+5.4}{-.4+16.2} = (-.40)$	$-\frac{-21.6}{+23.5-6.8} = (-.50)$	d ⎬ 2
$+.02$	$+.02$	$+.02$		$\Delta_2 M_0 = \frac{+.16}{8} = +.02$	$\Delta_2 Y_0 = \frac{-11.8}{42.7} = -.276$	e
$+1.12$	$+.28$	-1.12				f
-19.58	$+19.07$	-16.53	16.32			g ⎭
				$+5.6$	-33.4	a
$\theta_{pA}' = -14.5(1.52)3.16^2/19.58$				-11.4	$+45.6$	b
$\theta_{pB}' = +14.5(1.48+2.66)2.75^2/19.07$				$+23.7$	-23.7	b
$\theta_{pC}' = -14.5(2.34)0.21^2/16.53$				$-.1$	$-.4$	b
				$+17.8$	-11.9	c ⎫
				$-\frac{+.2}{+17.8+.4} = (-.24)$	$-\frac{-11.8}{-11.9-21.5} = (-.30)$	d ⎬ 3
$-.54$	$-.54$	$-.54$		$\Delta_3 M_0 = \frac{-4.3}{8} = -.54$	$\Delta_3 Y_0 = \frac{+3.6}{42.7} = +.083$	e
$-.33$	$-.08$	$+.33$				f
-20.45	$+18.45$	-16.74	17.04			g ⎭
				$+1.3$	-29.8	a
$\theta_{pA}' = -14.5(1.57)3.41^2/20.45$				-12.9	$+51.6$	b
$\theta_{pB}' = +14.5(1.43+2.60)1.41^2/18.45$				$+6.3$	-6.3	b
				-5.3	$+15.5$	c ⎫
				$-\frac{-4.3}{-5.3-17.8} = (-.20)$	$-\frac{+3.6}{+15.5+11.9} = (-.15)$	d ⎬ 4
$+.13$	$+.13$	$+.13$		$\Delta_4 M_0 = \frac{+1.06}{8} = +.13$	$\Delta_4 Y_0 = \frac{-2.3}{42.7} = -.054$	e
$+.22$	$+.05$	$-.22$				f
-20.10	$+18.63$	-16.83	16.75			g ⎭
				$+2.4$	-32.1	a
$\theta_{pA}' = -14.5(1.56)3.35^2/20.10$				-12.6	$+50.4$	b
$\theta_{pB}' = +14.5(1.44+2.63)1.88^2/18.63$				$+11.2$	-11.2	b
θ_{pC}' (negligible)				$+1.0$	$+7.1$	c ⎫
				$-\frac{+1.06}{1.0+3.3} = (-.15)$	$-\frac{-2.3}{7.1-13.3} = (-.25)$	d ⎬ 5
$-.02$	$-.02$	$-.02$		$\Delta_5 M_0 = \frac{-.15}{8} = -.02$	$\Delta_5 Y_0 = \frac{-1.8}{42.7} = -.042$	e
$+.17$	$+.04$	$-.17$				f
-19.95	$+18.65$	-17.02				g
				$+2.2$	-33.9	a
-23.43	$+17.58$	-14.08		$\Sigma(\Delta M_0) = +.27$	$\Sigma(\Delta Y_0) = -.796$	Check
$+.27$	$+.27$	$+.27$		$\times 8 = +2.2$	$\times 42.7 = -34.0$	
$+3.18$	$+.80$	-3.18				
-19.98	$+18.65$	-16.99				

Fig. 9 Example 1

for all potentially plastic regions. For the beam shown, such points would be A, B, and C. The computation of these elastic-solution moments is to the left and immediately above the double line.

Below the double line, the elasti-plastic redundant forces and beam-moments are approached to within three-figure accuracy by five successive corrective increments to the corresponding elastic-solution forces and moments. The two vertical columns at the right are used for calculating the angular and vertical deflections of the elastic center for each trial, and the three columns at the left are used for calculating the revised values of the moments at A, B, and C.

The lines of computations for each of the trials are lettered at the extreme right. The following comments, keyed to these letters, should explain the corresponding computations:

Line a. The components of movement of the elastic center, O, caused by elastic angle changes. Since superposition may be used for elastic deflections, each movement (angular or vertical) on this line is the summation of the elastic movement on line "a" of the previous trial and the increment of elastic movement found from the previous trial on line "e" of the previous trial. Thus, movement "5a" equals the sum of movements "4a" and "4e".

Lines b. The left-hand portion of these lines shows the computations for the plastic angles corresponding to the beam moments found in the previous trial, line "g", using, for all angles of this example, Eq. 14. The angle and the corresponding vertical displacement of the elastic center are tabulated to the right.

Line c. The summation of all movements, elastic and plastic, of the elastic center for the trial.

Line d. If the movements of line "c" are zero, the structure is solved. If not zero, this movement is an <u>approximate</u> measure of the increment of force which must be added to the elastic center for the next trial to make the movements zero. The adjective "approximate" is used because any new increments of force cause increments of movement from (1) elastic angle increments, which are easily determined from the previously determined elastic properties of the structure, and (2) plastic angle increments, which are non-linear and can only be determined from a complete recalculation of the plastic angles. Therefore, a fraction, r, of these movements equal to the expected ratio of elastic movement to total movement, will be restored to zero elastically. Line "d" shows the calculations of this expected ratio. Of course this ratio is difficult to predict for the trial $(n + 1)$ to be performed, but it may be calculated approximately from data furnished by the previous trials n and n - 1 as follows:

$$ r = - \frac{e_{n-1}}{c_n - c_{n-1}} \qquad (17) $$

For the first trial any value of r between, say, 0.10 and 0.60 may be assumed.

Line e (right). Line "c" multiplied by line "d", or the estimated elastic fraction of the restorative movement from the force increments for the next trial. Line "e" is then divided by line "f" (right), the elastic movement for a unit force, to determine the force increment. The force increments are shown in line "e" to the right of each vertical column.

Lines e, f, g, (left). Calculation of the revised values of beam moments resulting from the force increments of line "e" (right). These moments are

used to calculate the angles and movements for the next trial. For each trial the maximum of these moments is assumed to be M_u. Then this maximum moment is multiplied by the ratio M_e/M_u of the given M-ϕ relationship to obtain M_e for the trial.

Example 2—Design of a Two-Hinged Frame

Referring to Fig. 10 (a), the minimum bending strength required of constant-strength frame ABDE to support simultaneously the uniform load and concentrated load shown will be determined. The moment-curvature relationship shown in Fig. 7 will be used for the fram.

Computations required for the elastic solution of the frame are shown above the double line. As is customary, the single redundant, H_A, is found by allowing point A to move freely horizontally and dividing this displacement by the restorative displacement of a unit horizontal force at A. With H_A determined, the elastic-solution moments may be computed at all potential points of initial plasticity in the frame (points B, C, and D) as shown to the left and immediately above the double line.

Below the double line, elasti-plastic values of the redundant force and bending moments are approximated to within three-figure accuracy by four successfive corrective increments to the corresponding elastic-solution force and moments. The vertical column at the right is used for calculating the horizontal displacement of A for each trial, and the three columns at the left are used for calculating the revised values of the moments at B, C, and D. The lines of computations for each of the trials are lettered at the extreme right. An explanation corresponding to the letters of any of these lines may be found in the comments following Example 1.

One additional explanatory comment for line "b" is that Eqs. 14 and 12 are used, with but slight theoretical error, for the calculation of the plastic angles in member BD at D, C, and B, instead of the more accurate Eqs. 16 and 10. The results of a solution based on Eqs. 16 and 10 are shown on the last line of Fig. 10. The error in moments for this example is about 1%, which is negligible.

Example 3—Design of a Fixed-Ended Frame

Referring to Fig. 11, the minimum bending strength required of constant-strength frame ABCDE to support the 10 kip load shown will be determined. The moment-curvature relationship shown in Fig. 7 will be used for the frame.

Computations required for the elastic solution are shown above the double line of Fig. 11. Indeterminate to the third degree, the frame may be cut at A and the three forces at 0 necessary to restore rigid arm OA to its required position are calculated. Using these forces and the load, the elastic-solution moments are computed for all points at which plasticity may initiate, that is, points of probable maximum moment for the region near the point. Computations for these moments are shown at the left and immediately above the double line of Fig. 11.

Below the double line, elasti-plastic values of the three redundant forces and the bending moments are approximated by three successive corrective increments to the corresponding elastic-solution forces and moments. Three vertical columns at the right are used for calculating the rotational, vertical, and horizontal movements of O for each trial, and the five vertical columns at the left are used for calculating the revised values of moments at E, D, F, B, and A. An explanation for any line of computations may be found by referring
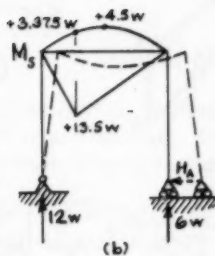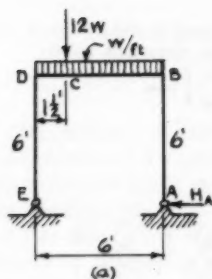
Sign Convention: $\overline{DB}$ $\underline{\quad A}$ $_+$

	S	$\overline{y}_A$	$\int y_A^2 ds$	$\int y_A M_s ds$
AB	6	3	54 + 18	0
BD	6	6	216	6(18+40.5)w
DE	6	3	54 + 18	0
Σ	18		360	$\Delta_A' = 351w$

M_s +3.375w   +4.5w

+13.5w

12w   6w

(b)

$H_A = -\dfrac{351w}{360} = -0.975w$

M_D ($y_A = +6$)	M_C (+6)	M_B (+6)	M_e
0	+16.87(w)	0	
-5.85	-5.85	-5.85	
-5.85	+11.02	-5.85	9.18

$y_A\theta' =$ Δ_A'

$L_c = M_c(1/10.5 + 1/1.5) = .762M_C$ $\qquad$ 0 $\qquad$ a

$\theta_{pC}' = +14.5(.762)(11.02)1.84^2/11.02 = +37.2$ $\qquad$ +223.2 $\qquad$ b,c

$r = (-.50)$ (assume) $\qquad$ d

$\Delta_1 H_A = \dfrac{-111.6}{360} = -.311(w)$ $\qquad$ e,f

-1.87	-1.87	-1.87	
-7.72	+9.15	-7.72	7.62

$\qquad$ -111.6 $\qquad$ a

$\theta_{pC}' = +14.5(.762)(9.15)1.53^2/9.15 = +25.8$ $\qquad$ +154.8 $\qquad$ b

$\theta_{pD}' = -14.5(7.72/12 + 6).10^2/7.72 = -0.1$ $\qquad$ -.6 $\qquad$ b

$\theta_{pB}' = -14.5(7.72/6 + 6).10^2/7.72 = -0.1$ $\qquad$ -.6 $\qquad$ b

$\qquad$ +42.0 $\qquad$ c

$\dfrac{-111.6}{+42-223.2} = (-.60)$ $\qquad$ d

$\Delta_2 H_A = \dfrac{-25.2}{360} = -.070$ $\qquad$ e,f

-.42	-.42	-.42	
-8.14	+8.73	-8.14	7.28

$\qquad$ -136.8 $\qquad$ a

$\theta_{pC}' = +14.5(.762)(8.73)1.45^2/8.73 = +23.2$ $\qquad$ +139.2 $\qquad$ b

$\theta_{pD}' = -14.5(8.14/12 + 6).86^2/8.14 = -8.8$ $\qquad$ -52.8 $\qquad$ b

$\theta_{pB}' = -14.5(8.14/6 + 6).86^2/8.14 = -9.7$ $\qquad$ -58.2 $\qquad$ b

$\qquad$ -108.6 $\qquad$ c

$\dfrac{-25.2}{-108.6-42} = (-.167)$ $\qquad$ d

$\Delta_3 H_A = \dfrac{+18.1}{360} = +.050$ $\qquad$ e,f

+.30	+.30	+.30	
-7.84	+9.03	-7.84	7.52

$\qquad$ -118.7 $\qquad$ a

$\theta_{pC}' = +14.5(.762)(9.03)1.51^2/9.03 = +25.2$ $\qquad$ +151.2 $\qquad$ b

$\theta_{pD}' = -14.5(7.84/12 - 6).32^2/7.84 = -1.3$ $\qquad$ -7.8 $\qquad$ b

$\theta_{pB}' = -14.5(7.84/6 - 6).32^2/7.84 = -1.4$ $\qquad$ -8.4 $\qquad$ b

$\qquad$ +16.3 $\qquad$ c

$\dfrac{+18.1}{+16.3+108.6} = (-.14)$ $\qquad$ d

$\Delta_4 H_A = \dfrac{-2.3}{360} = -.006$ $\qquad$ e

-.04	-.04	-.04	
-7.88w	+8.99w	-7.88w	7.48

-5.85	+11.02	-5.85	
-2.02	-2.02	-2.02	
-7.87w	+9.00w	-7.87w	

-121.0 $\qquad$ a

$\Sigma(\Delta H_A) = -.337w$ $\qquad$ Check

$\times 360 = -121.2w$

-7.82w $\quad$ +9.05w $\quad$ -7.82w $\qquad$ Using "exact" Eqs. 16, 10.

Fig. 10 Example 2
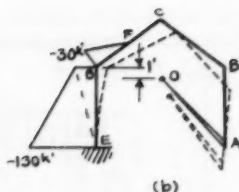
851-17

Fig. 11 (a) and (b) structural frame diagrams

	S	x_o	y_o	$\int x_o^2 ds$	$\int y_o^2 ds$	$\theta_o'=$ $\int M_s ds$	$\Delta_o'=$ $\int x_o M_s ds$	$\Delta_o'=$ $\int y_o M_s ds$
AB	10	+8	-4	640	160 + 83.3			
BC	10	+4	+4	160 + 53.3	160 + 30			
CD	10	-4	+4	160 + 53.3	160 + 30	-75	+500	-150
DE	10	-8	-4	640	160 + 83.3	-150	+1200	+350
						-650	+5200	+3680
Σ	40			1706.6	866.7	-875	+6900	+3880

$$M_o = -\frac{-875}{40} = +21.88^{k'}$$

$$Y_o = -\frac{+6900}{1707} = -4.04^k$$

$$X_o = -\frac{+3880}{867} = -4.48^k$$

M_E $(x_o=-9)$ $(x_o=-8)$	M_D (-6)	M_F (-4)	M_B $(+1)$ $(+6)$	M_A (-9) $(+8)$	M_c		θ_o'	ΔM_o	Δ_o'	ΔX_o	Δ_o'	ΔY_o	
-130.0	-30.0	0	0	0									
+21.9	+21.9	+21.9	+21.9	+21.9									
+40.3	-4.5	-17.9	-4.5	+40.3									
+32.3	+32.3	+16.1	-32.3	-32.3									
-35.5	+19.7	+20.1	-14.9	+29.9	29.6		-92.0		+828		+736		b
$\theta_{pE}'=-14.5(6.5)5.9^2/35.5$							+.4		-4		+3		b
$\theta_{pA}'=+14.5(6.7)0.3^2/29.9$							-91.6		+824		+739		c
						(assume)	(-.25)		(-.25)		(-.25)		d ⟩1
+.6	+.6	+.6	+.6	+.6			$\frac{+22.9}{40}$=+.57		$\frac{-206}{867}$=-.24		$\frac{-185}{1707}$=-.11		e
+2.2	-.2	-1.0	-.2	+2.2									f
+.9	+.9	+.4	-.9	-.9									g
-31.8	+21.0	+20.1	-15.4	+31.8	26.5								g
							+22.9		-206		-185		a
$\theta_{pE}'=-14.5(6.0)5.3^2/31.8$							-78.0		+702		+624		b
$\theta_{pA}'=+14.5(6.8)6.1^2/31.8$							+88.0		-792		+700		b
							+32.9		-296		+1143		c
						$\frac{+22.9}{32.9-32.9}$=(-.20)		$\frac{-206}{214-824}$=(-.18)		$\frac{-185}{1143-729}$=(-.50)		d ⟩2	
-.2	-.2	-.2	-.2	-.2			$\frac{-6.6}{40}$=-.17		$\frac{+54}{867}$=+.06		$\frac{-571}{1707}$=-.34		e
-.5	+.1	+.2	+.1	-.5									f
+2.7	+2.7	+1.4	-2.7	-2.7									g
-29.8	+23.6	+21.5	-18.2	+28.4	24.8								g
							+16.3		-152		-756		a
$\theta_{pE}'=-14.5(5.6)5.0^2/29.8$							-68.0		+612		+544		b
$\theta_{pA}'=+14.5(6.2)3.6^2/28.4$							+41.0		-369		+328		b
							-10.7		+91		+116		c
						$\frac{-6.6}{-10.7-32.9}$=(-.15)		$\frac{+54}{91+296}$=(.14)		$\frac{-571}{116-1143}$=(.56)		d ⟩3	
0	0	0	0	0			$\frac{+1.6}{40}$=+.05		$\frac{-13}{867}$=-.01		$\frac{-65}{1707}$=-.04		e
+.1	-.0	-.1	-.0	+.1									f
+.3	+.3	+.2	-.3	-.3									g
-29.4	+23.9	+21.6	-18.5	+28.2									g
-35.5	+19.7	+20.1	-14.9	+29.9			+17.9		-165		-821		a
+.4	+.4	+.4	+.4	+.4			$\Sigma(\Delta M_o)=$ +.45		$\Sigma(\Delta X_o)=$-.19		$\Sigma(\Delta Y_o)=$-.49		⟩Check
+1.8	-.2	-.8	-.2	+1.8			×40=+18.0		×867=-165		×1707=-836		
+3.9	+3.9	+2.0	-3.9	-3.9									
-29.4	+23.8	+21.7	-18.6	+28.2									

Fig. 11 Example 3

to the explanation included in Example 1 corresponding to the letter at the extreme right of the line.

Design vs. Analysis

Examples 1-3 are "designs" in that they indicate the flexural strength requirements of a structure and loading. This use of "design" is admittedly very narrow since the many other requirements of a satisfactory design are not considered.

The other type of problem of practical importance involving single-span beams or frames is the analysis of a structure with a given M-ϕ relationship and ultimate bending strength, M_u, for a given loading, P'. Since elasti-plastic moments, unlike the conventional elastically-determined moments, are not proportional to load, the usual "analysis", that is, determination of the moments in a given structure for a given load, has little significance in determining reserve strength (although such moments may be readily determined by a slight modification of the procedures of Examples 1-3.) The best measure of reserve strength of a structure is the ratio of the collapse load, P_c, to the working load, P', or load factor (P_c/P'). The collapse load may be found by simple proportion from the results of any design solution, such as Examples 1-3, in which any convenient magnitude of load, P, is assumed, and a corresponding required strength, M_u, is obtained: Thus:

$$P_c = \frac{M_u'}{M_u} P \qquad (18)$$

from which by definition,

$$Load\ Factor = \frac{M_u' P}{M_u P'} \qquad (19)$$

Comments on the Procedure

Certain characteristics of the general procedure presented are illustrated by Examples 1 to 3 and should be discussed:

1. Scope. This general procedure applies to all single cell structures to which the elastic center or column analogy methods apply for elastic analysis.

2. Accuracy and Convergence. An accuracy sufficient for most practical purposes is usually achieved by only three increments of force and moment, as shown by examples 1 and 2. More cycles may be needed for moment-curvature relationships with a plastic area larger than that of Fig. 7, that is, relationships with larger values of k or M_u/M_e. In general more plasticity induces more plastic angles and retards convergence.

The ratio r is usually estimated, using Eq. 17, on the basis of quantities of the previous increment. This lag may cause a temporary reduction in convergence, or even divergence, especially in increments in which new plastic angles appear. With experience it is possible to anticipate these factors, arbitrarily adjust r accordingly, and hasten convergence. Any increment which causes general divergence should be discarded, although the quantities of the increment should be used to calculate improved values of r for the replacement increment.

Fortunately, the lag in the value of ratio r as given by Eq. 17 becomes smaller, and convergence becomes faster, as increments become smaller. Therefore, in any problem there will be a terminal increment for which the computed value of r is "exact" relative to the accuracy of the preceding computations, and sudden complete convergence will occur. Since the benefits of complete convergence are more psychological than practical, Examples 1-3 have not been continued to this point.

3. Checking. As shown by the examples, a complete check of an elasti-plastic solution involves the following relatively simple steps: (a) check only the last computation of all plastic angles and the deflection components resulting therefrom; (b) add all increments of the forces acting on the elastic center; (c) from these forces calculate the corresponding elastic deflections, which should check the elastic deflection on line "a" following the last force increment; and (d) also from these forces calculate total change in moments which added to the elastic-solution moments should check the final moments.

Calculation of Deflections

Deflections may be readily calculated from the angle changes calculated in an elasti-plastic solution by using the following well-known geometric principle: the horizontal (or vertical) displacement of a point J on the axis of a member with respect to a frame of reference attached to another point A on the axis equals the first moment, about a horizontal (or vertical) line through J, of all angle changes between J and A.

Then, (remembering that $d\theta'_e = Mds$) the components of deflection of any point J on a fixed ended beam or frame would be:

$$\frac{M_e}{\phi_e}\Delta = \int_J^A x_J\, d\theta' = \sum_J^A (x_J\, \theta'_p) + \int_J^A x_J\, Mds \qquad (20)$$

$$\frac{M_e}{\phi_e}\Delta = \int_J^A y_J\, d\theta' = \sum_J^A (y_J\, \theta'_p) + \int_J^A y_J\, Mds \qquad (21)$$

Also, the rotation of the axis at J will obviously be:

$$\frac{M_e}{\phi_e}\theta_J = \int_J^A d\theta' = \sum_J^A \theta_p + \int_J^A Mds \qquad (22)$$

where A is the fixed end nearest to J.

These expressions may be used for a frame or beam hinged or partly fixed at a support A if they include the angle of rotation of the member at A as an angle change.

Note that the accuracy of these deflections depends on the absolute accuracy of the angle changes from which they are computed. This accuracy is therefore inferior to the accuracy of elasti-plastically computed moments, which depends only on the relative accuracy of the angle changes.

Table II shows the relative values of deflections before collapse and at the elastic limit for the structures of Examples 1 to 3. As would be expected, above the elastic limit deflection increases at a higher rate than load, but not excessively so for these examples.

	Ratio, collapse load to load at elastic limit	Ratio, deflection before collapse to deflection at elastic limit	
		Vertical	Horizontal
Beam, Example 1	1.41	2.24	--
Frame, Example 2	1.48	2.74	4.78
Frame, Example 3	1.45	1.89	2.12

Table II Deflections at concentrated load of example structures

Quantitative Comparison with the Orthodox Methods

Figs. 12-14 show comparisons of the strengths of various structures as given by the elasti-plastic, the elastic, and the plastic theories. Note that all values are based on an elasti-plastic moment-curvature relationship with $k = 29$ and $M_u = 1.2M_e$. For different moment-curvature relationships the relative strengths given by the elastic and the plastic theories would obviously be different; should k increase and approach infinity, the plastic theory values would approach unity, and should k decrease and approach zero, the elastic theory values would approach unity.

These comparisons tend to verify the claim of Table I—that neither the elastic theory nor the plastic theory accurately indicate the collapse load of a constant strength structure made of the usual structural material of appreciable but finite plasticity. They also show the error in the proposal that an average of the elastic and plastic collapse loads approximates the true collapse load.[22]

Inherent Limitations of Elasti-Plastic Solutions

Besides its obvious irrelevance for structures which fail by permanent set or fatigue, the method herein presented is inherently limited by its neglect of two types of factor.

First, the method presented neglects axial and shear deformations. Although these deformations are non-rotational, they cause deflections which should be added to those calculated from curvature, and, if they are neglected, accuracy is correspondingly reduced. As is well known, the loss in accuracy in an elastic solution from neglecting these deformations is almost always insignificant, and the author can see little reason why the relative importance of these deformations should increase in the plastic range. However, little is known about such deformations when they occur in combination with flexural deformations. If such deformations were known in quantity, and deemed important, the method herein presented could be expanded with little difficulty to account for them.

The second inherent limitation of the method is that it is "non-historic"— the increments of force and deformation which it describes cannot represent the actual behavior of the structure with increasing time as the loads are applied (because throughout the solution the loads are assumed to be constant). Obviously, under conditions for which a force-deformation relationship can only be defined in historic terms a non-historic method becomes inaccurate.

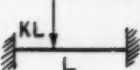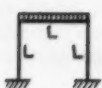The moment-curvature relationship becomes a function of the loading
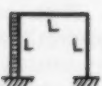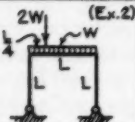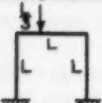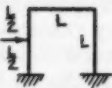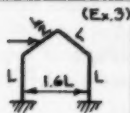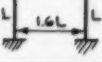
Structure (I is constant)	Relative Strength by:				Structure (I is constant)
	Elasti-Plastic k=29, M_e=.83M_u	Elastic Max.M=M_u	Plastic M_y=M_e	Plastic M_y=M_u	
	1.0	.893	.993	1.190	
	SEE FIG. 13				
	1.0	.903	.837	1.005	
	SEE FIG. 14				
	1.0	.820	1.195	1.435	(Ex.2)
	1.0	.816	.888	1.078	
	1.0	.845	.928	1.113	
	1.0	.800	1.120	1.342	
	1.0	.891	.972	1.167	(Ex.3)
	1.0	.829	.980	1.176	

Fig. 12 Strengths of structures by various theories

history as soon as any increase in loading causes simultaneous increases and reductions in plastic-range moments at various sections. As is well known, a reduction causes curvature to decrease at an elastic rather than elasti-plastic rate, and the assumed moment-curvature relationship is violated. Obviously, a moving loading usually causes such moment reductions, and variable fixed-position loads applied in certain sequences may cause such moment reductions. Therefore, an elasti-plastic solution for either of these loading types usually has to be historic and be based on time increments. Such a solution could be considered as a series of subsidiary solutions, one for each time increment, and each subsidiary solution could be similar to the solutions presented herein. The complete solution for such loading types is obviously almost prohibitively long.

Simultaneous increases and reductions in moments with increasing load also occurs in a structure as soon as the curvature at the section of maximum moment in any plastic region exceeds ϕ_u, Fig. 3(a). As ϕ_u is exceeded, the moment at the section becomes smaller than M_u, and, since the shears in the member must be maintained to satisfy statics, all moments throughout the plastic region are similarly reduced. However, further consideration shows
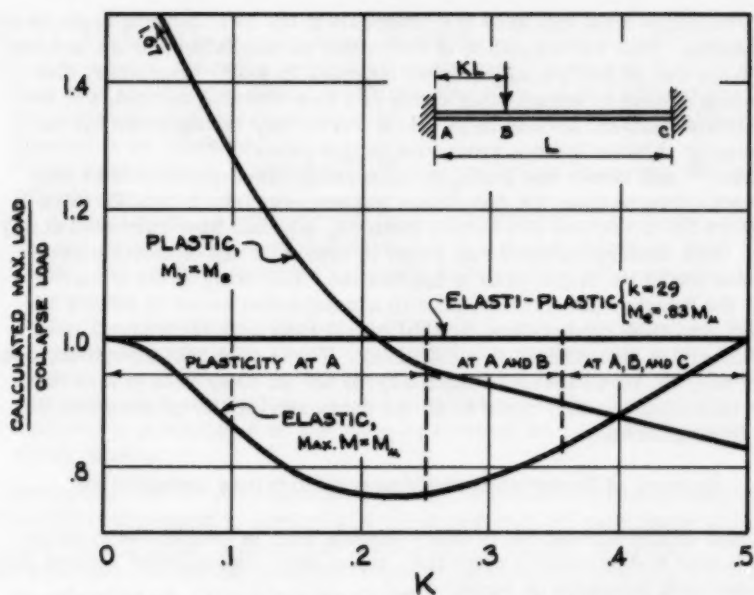
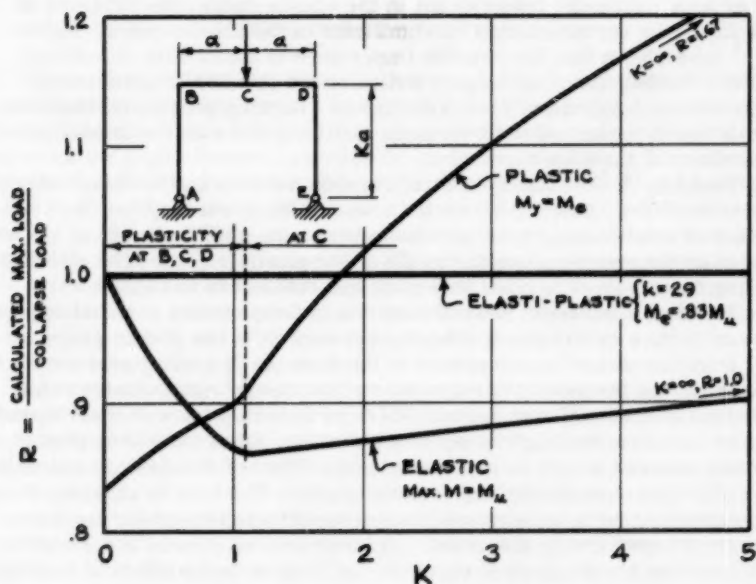Fig. 13   Comparison of theories for fixed-ended beam

Fig. 14   Comparison of theories for two-hinged frame

the unimportance of this phenomenon. These moment reductions reduce curvature at all sections except the section of maximum moment. At this

section curvature must increase tremendously if the total bending angle is not to be reduced. This concentration of curvature causes failure at the section before there can be further appreciable increase in load. Therefore, this phenomenon causes no appreciable error in a non-historic method, and the post-ultimate-moment portion of the M-ϕ curve may be neglected for an elasti-plastic solution (as has been done in this paper).

Drucker[23] has shown that a single, increasing, fixed-position load may cause moment reductions for non-linear moment-curvature relationships even before the curvature at ultimate moment, ϕ_u, has been exceeded at any section. This demonstration would seem to deprive a non-historic elasti-plastic method of its major area of application. However, in the writer's opinion, the Drucker phenomenon (a) will almost never occur in beams and frames of practical proportions, and (b) would have relatively small effect on the magnitude of moments if it did occur. These opinions, almost impossible to prove, should be treated skeptically, if for no other reason than that their verification can only come from the continued futility of sustained attempts to dispose them.

Sources of Variation in the Moment-Curvature Relationship

Although moment about the primary bending axis is usually the most important factor in determining curvature, many other "secondary" factors may have significant influence on curvature.

A list of such factors could include:

a. Axial force, shear, and moments about axes other than the primary bending axis. Although insignificant in the elastic range, the influence of these forces on curvature may be significant in the plastic range. Ketter et al[14] have shown that the relative importance of elastically-calculated axial and bending stresses roughly indicates the relative importance of modifications in curvature from axial force. Lacking precise information, it would seem conservative to estimate similarly the relative importance of the effect of shear on curvature.

b. Buckling[14, 24] which, in turn, depends not only on the stress-strain properties of the material, but on the shape of the cross-section, the amounts of axial force, shear, and secondary moments, the external restraints on the member, and the length of the plastic region. The effect of buckling on curvature is often important and difficult to evaluate.

c. Residual stresses[14] and differentials in temperature and shrinkage. All usually have rather small effect on curvatures in the plastic range.

d. Duration of loading. Increase in the duration of loading evidently tends to reduce the plastic curvature area for steel[13] and increase it for reinforced concrete,[18] and the true M-ϕ relationship for a section depends upon the complete loading history of the section. Since the elasti-plastic method presented herein is non-historic, the effect of this history can only be roughly approximated by any assumed curve. The loss in accuracy from this approximation is usually relatively insignificant (except for the unusual conditions already discussed). A more serious loss in accuracy results from the present lack of experimental data as to the effect of loading-time on curvature.

e. Sense of the bending for all members with cross section not symmetrical about the neutral axis and made of materials which have different stress-strain properties in tension and compression, for example, most reinforced concrete members.

f. Variation in reinforcement in reinforced concrete. Although the gross area of concrete, neglecting reinforcement, may be a sufficiently accurate measure of elastic stiffness, the effect of variation in the amount and placement of the reinforcement on the M-ϕ curve is usually too important to be neglected.

g. Variations in a material[25,26] and variations in as-built structural dimensions.

h. Restraints of adjacent structural components. Qualitative analysis usually shows that if such restraints on a primary structure must be neglected, the solution is on the safe side as far as the primary structure is concerned.

i. The transition from ductile to brittle fracture for steel.[27] Large size, low temperature, stress concentrations, and certain metallurgical factors tend to encourage this transition, which involves a sudden, rather complete, loss of plasticity. Brittle fracture must be prevented by the customary precautions or the structure cannot be considered as being elasti-plastic.

Obviously, the M-ϕ curves used in an elasti-plastic solution may be adjusted from section to section of a structure with all necessary or possible refinement for those of these factors which are deemed important. The solution will then account for such factors with an accuracy equal to the accuracy of the adjustments. The difficulty is that our present knowledge of these factors is such that often these adjustments cannot be made with the required degree of accuracy.

Elasti-Plastic Design Precautions

Also, unfortunately, the required degree of accuracy is higher than is superficially apparent. A very important principle in applying elasti-plastic methods to structural design is that an overestimation of plastic strength causes a disproportionate reduction in the factor of safety. For example, assume that the square frame with horizontal load applied to mid-height of one column, Fig. 12, has been designed using the elasti-plastic method, with $k = 29$ and $M_e = 0.83 M_u$. Then, if unforeseen factors bring on failure when the maximum moment in the frame is 8.3% less than the design maximum (M_e and k remaining the same) the load at failure will be 20.9% less than the design load. Or if unforeseen factors should cause failure at a maximum moment of M_e, which is 16.7% less than the design maximum and which would eliminate all plastic action, the loss in strength would be about 33.3%.

It is evident from these examples that special care must be taken not to overestimate the plastic area of the M-ϕ relationship. It would also seem that the simple load-factor method of providing structural safety for elastic design should be modified and made more complex to account accurately for this sensitivity of strength with plasticity. Until careful studies have determined these modifications, the author would suggest that an initially conservative M-ϕ relationship be modified by a 50% reduction in "k", Fig. 4(a), and then used with the standard load factors to provide for this sensitivity in a design.

If an elasti-plastic method is to be substituted in a code for the elastic method under which the code evolved, it must be remembered that the true safety built into a code depends not only on its allowable stresses, but on its specified loads, its theory for stress distribution at a cross section, and its theory for force distribution throughout the structure. Thus, a code may

compensate for an overly safe theory for force-distribution by an abnormally high value of allowable stress or by a low value of specified load. Obviously, if the overly conservative elastic theory of a code is replaced by a more realistic elasti-plastic theory, all of these hidden compensations must be found and eliminated. For example, the 20% increase in stress allowed by the A.I.S.C. steel building specifications (Sect. 15 (a))[28] at the supports of continuous beams should probably not be used with an elasti-plastic theory of moment distribution. On the other hand, few such compensations need be eliminated from a code which has been developed for structures of variable $\underline{I}$ since the elastic and the elasti-plastic theories are almost equivalent for such structures.

Thus, the greatest potential value of elasti-plastic methods is obviously not simplicity. Nor is it structural economy; the material savings in a flex-ural member will be small at best—and probably negligible relative to cost of the entire structure. Nor is it universality; elastic methods will always be best for many design requirements. Elasti-plastic methods are potentially most valuable as a basis for a better understanding of structural action, a better correlation of research into flexural behavior, a proportioning of mem-bers (especially reinforced concrete) for ductility as well as strength, clearer and more realistic specifications for certain types of design, and a compro-mise between the extremes now offered by elastic and plastic analyses.

CONCLUSIONS

1. Both the elastic and plastic portions of the M-ϕ relationship as well as the statics of flexure must be used to obtain accurately the collapse load for flexural structures of ordinary materials.

2. The curvatures of the M-ϕ diagram which exceed curvature at the point of maximum moment M_u should be neglected for safety.

3. The remainder of the M-ϕ diagram can be approximated accurately enough for most engineering purposes by two segments, one defining the elas-tic and the other the plastic portion of the diagram.

4. For the common moment distributions, the total plastic rotation on a plastic region may be accurately calculated from a simple formula, and for acceptable accuracy the rotation angle may usually be considered as concen-trated at the point of maximum moment of the region.

5. Slight extension of the well-known techniques of elastic analysis in-volving use of the elastic center and a successive approximation technique, as well as use of a calculated predicted ratio of elastic to total deflection, provide a practical method of elasti-plastic solution. Superposition cannot be used for plastic deformations and deflections.

6. Achievement of the required accuracy in an elasti-plastic solution of an actual structure may require evaluation of the effect of several "secondary" factors on curvature.

7. Although usually not important, the effect of axial and shear (non-rota-tional) deformations may be included in an elasti-plastic solution by extension of the method here presented.

8. If the M-ϕ diagram is a function of time for a loading, the inherent error of the non-historic method here presented (which may or may not be significant) may be avoided only by use of a historic method.

9. An elasti-plastic solution lends itself to the calculation of deflections, provided the absolute values of the curvatures are known.

10. The hypersensitivity of elasti-plastically-determined structural strength to variation in the plasticity of a material indicates the need for special techniques to determine load factors which will preserve structural safety. Also, other precautions are necessary if an elasti-plastic method is to be used with specifications which have evolved in the context of other methods.

NOTATION
(in order of appearance in the paper)

M Bending moment

ϕ Curvature, or angle change per unit length

M_y Limit or yield moment of the plastic theory

M_u Ultimate bending moment

M_e Elastic limit bending moment

ϕ_u Curvature at ultimate bending moment

ϕ_e Elastic limit curvature

V Resultant shearing force on a section of a beam

S_L, S_R Lengths of plastic region to left and right, respectively, of a concentrated load

θ_E Elastic portion of angle of rotation of beam-axis at a point

M_m Maximum bending moment in a plastic region ($M_e < M_m \leq M_u$)

θ_p Plastic portion of rotation of axis at one end of a plastic region with respect to axis at other end of the plastic region—the plastic angle

ϕ_p Plastic portion of curvature

ϕ_E Elastic portion of curvature

S_p Length of plastic region

k ϕ-intercept of elasti-plastic line of M-ϕ diagram divided by ϕ_e (See Fig. 5(a))

L Distance between points of zero moment of a two-segment moment diagram (See Fig. 5(c))

w Uniformly distributed transverse load per unit of length of beam

θ_{pR}, θ_{pL} θ_p for plastic regions to right and left, respectively, of a concentrated load

θ' An angle of rotation, θ, multiplied by the ratio M_e/ϕ_e. In general, the superscript prime ($'$) applied to a component of movement represents the factor M_e/ϕ_e.

M_s Bending moment in statically determinate structure

x_O, y_O Abscissa and ordinate, respectively, for point on beam-axis using coordinate axes with origin at elastic center O.

$\bar{x}$, $\bar{y}$ Abscissa and ordinate, respectively, to a centroid

θ_O, Δ_O, δ_O Angle of rotation, vertical displacement, and horizontal displacement, respectively, of elastic center O

M_O, Y_O, X_O Moment, vertical force, and horizontal force, respectively, at elastic center O.

n Number of trial of an elasti-plastic solution

$\Delta_n M_O$ Increment in M_O for trial n

r Predicted ratio of elastic movement to total movement of elastic center O for a trial (Eq. 17)

P' Working load on a structure

M_u' Actual ultimate bending strength of a structure

P_c Load causing collapse

REFERENCES

1. "Recent Progress in the Plastic Methods of Structural Analysis," Symonds, P. S., and Neal, B. G., Journal of the Franklin Institute, Vol. 252, Nos. 5, 6, (Nov., Dec. 1951), pp. 383-407, 468-492.

2. "Theory of Limit Design," Van den Broek, J. A., Transactions, ASCE, Vol. 105 (1940), pp. 638-730.

3. "Theory of Inelastic Bending with Reference to Limit Design," Hrennikoff, A., Transactions ASCE, Vol. 113 (1948), pp. 213-268.

4. "Plastic Strength and Deflections of Continuous Beams," Knudsen, K. E., Yang, C. H., Johnston, B. G., and Beedle, L. S., Welding Journal, Vol. 32, No. 5 (May 1953), pp. 240s-256s.

5. "The Interpretation of Failure Loads in the Plastic Theory of Continuous Beams and Frames," Symonds, P. S. and Neal, B. G., Journal of the Aeronautical Sciences, Vol. 19, No. 1, pp. 15-22 (Jan. 1952).

6. "Ultimate Slopes and Deflections—A Brief for Limit Design," Ernst, G. C., Proceedings ASCE, Separate 583, Vol. 81 (Jan. 1955).

7. "Strength of Materials," Part II, Timoshenko, S., Van Nostrand, New York, 1941.

8. "Theory of Limit Design," Winter, G., Discussion, Transactions ASCE, No. 105 (1940), pp. 673-679.

9. "Plastic Behavior of Wide-Flange Beams," Luxion, W. W. and Johnston, B. G., Progress Report No. 1, The Welding Journal, Vol. 27, No. 11, (1948), pp. 538s-554s.

10. "Connections for Welded Continuous Portal Frames," Topractsoglou, A. A., Beedle, L. S. and Johnston, B. G., Part I, Welding Journal, Vol. 30, No. 7 (1951) pp. 359s-384s; Part II, Vol. 30, No. 8 (1951) pp. 397s-405s.

11. Residual Stress and the Yield Strength of Steel Beams," Yang, C. H., Beedle, L. S., and Johnston, B. G., Progress Report No. 5, The Welding Journal, Vol. 31, No. 4 (1952), pp. 205s-229s.

12. "Moment-Rotation Characteristics of Column Anchorages," Salmon, C. G., Schenker, L., and Johnston, B. G., Proceedings ASCE, Vol. 81, Separate 660, (April 1955).

13. "Inelastic Behavior of Ductile Members under Dead Loading," Clark, M. E., Corten, H. T., Sidebottom, O. M., University of Illinois Experiment Station Bulletin No. 426, (October 1954), Urbana, Ill.

14. "Plastic Deformation of Wide-Flange Beam-Columns," Ketter, R. L., Kaminsky, E. L., and Beedle, L. S., Proceedings ASCE, Separate No. 330, (November 1953).

15. "Elastic-Plastic Theory of Bending in Reinforced Concrete Beams (Théorie Elasto-plastique de la Flexion dans les poutres en Béton Armé)," Chambaud, Annales de l'Institut Technique du Bâtiment et des Travaux Publics, No. 101 Béton, Béton Armé No. 10, (Nov. 1949).

16. "A Study of Combined Bending and Axial Load in Reinforced Concrete Members," Hognestad, E., University of Illinois Experiment Station Bulletin No. 399, (Nov. 1951), Urbana, Ill., pp. 43-54.

17. "Inelastic Behavior of Reinforced Concrete Members Subjected to Short-Time Static Loads," Lee, L. H. N., Proceedings ASCE, Separate 286, Vol. 79 (Sept. 1953).

18. "Behavior of Under-Reinforced Concrete Beams under Long-Term Loads," Bulletin, University of Connecticut Engineering Experiment Station (to be published).

19. "Analysis of Continuous Frames by Distributing Fixed End Moments," Cross, H., Transactions ASCE, Vol. 96 (1932), pp. 1-156.

20. "Elastic Arch Bridges," McCullough and Thayer, Wiley, New York, (1931), Chapter IV.

21. "The Column Analogy," Cross, H., University of Illinois Engineering Experiment Station, Bulletin No. 215 (October 1930), Urbana, Ill.

22. "The Inelastic Behavior of Engineering Materials and Structures," Freudenthal, A. M., Wiley, New York, (1950) pp. 499.

23. "Plasticity of Metals—Mathematical Theory and Structural Applications," Drucker, D. C., Proceedings ASCE, Vol. 76, Separate 27 (August 1950).

24. "Flange Buckling, A Limitation of Plastic Design of Beams, Frames and Columns," Winter, G., Fourth International Congress of Bridge and Structural Eng'g., Cambridge 1952, Final Report, Zurich (1953), pp. 139.

25. "An Evaluation of Plastic Analysis as Applied to Structural Design," Johnston, B. G., Yang, C. H., and Beedle, L. S., Progress Report No. 8, Welding Journal, Vol. 32, No. 5 (May 1953), pp. 224s-239s.

26. "Safety and the Probability of Structural Failure," Freudenthal, A. M. Proceedings ASCE, Separate 468, Vol. 80 (August 1954).

27. "Some Research Activities Related to Welded Structures," Jonassen, F., Engineering Experiment Station News, The Ohio State University, Vol. 22, No. 3 (June 1950), pp. 11-17, 38-42.

28. "Steel Construction Manual," AISC, New York, 5th Ed., (1947) p. 286.

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